Basic Mathematics

The Laplacian

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The aim of this package is to provide a short self assessment programme for students who want to apply the Laplacian operator.

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The full range of these packages and some instructions, should they be required, can be obtained from our web page Mathematics Support Materials.

1. Introduction (Grad, Div, Curl)

The vector differential operator $\boldsymbol{\nabla}$, called "del" or "nabla", is defined in three dimensions to be:

$$\boldsymbol{\nabla} = \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}$$

The result of applying this vector operator to a scalar field is called the **gradient of the scalar field**:

$$\mathrm{grad} f(x,y,z) = \boldsymbol{\nabla} f(x,y,z) = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}\,.$$

(See the package on **Gradients and Directional Derivatives**.)

The *scalar product* of this vector operator with a vector field $\boldsymbol{F}(x,y,z)$ is called the **divergence of the vector field**:

$$\mathrm{div}\boldsymbol{F}(x,y,z) = \boldsymbol{\nabla}\cdot\boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\,.$$

The *vector product* of the vector $\boldsymbol{\nabla}$ with a vector field $\boldsymbol{F}(x, y, z)$ is the **curl of the vector field**. It is written as $\text{curl}\,\boldsymbol{F}(x, y, z) = \boldsymbol{\nabla} \times \boldsymbol{F}$

$$\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{F}(x, y, z) &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\boldsymbol{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\boldsymbol{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k} \\
&= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix},
\end{aligned}$$

where the last line is a formal representation of the line above. (See also the package on **Divergence and Curl**.)

Here are some revision exercises.

EXERCISE 1. For $f = x^2y - z$ and $\boldsymbol{F} = x\boldsymbol{i} - xy\boldsymbol{j} + z^2\boldsymbol{k}$ calculate the following (click on the **green** letters for the solutions).

(a) $\boldsymbol{\nabla} f$ (b) $\boldsymbol{\nabla} \cdot \boldsymbol{F}$

(c) $\boldsymbol{\nabla} \times \boldsymbol{F}$ (d) $\boldsymbol{\nabla} f - \boldsymbol{\nabla} \times \boldsymbol{F}$

2. The Laplacian

The **Laplacian** operator is defined as:

$$\boxed{\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\,.}$$

The Laplacian is a scalar operator. If it is applied to a scalar field, it generates a scalar field.

Example 1 The Laplacian of the scalar field $f(x,y,z) = xy^2 + z^3$ is:

$$\begin{aligned}
\nabla^2 f(x,y,z) &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\
&= \frac{\partial^2}{\partial x^2}(xy^2 + z^3) + \frac{\partial^2}{\partial y^2}(xy^2 + z^3) + \frac{\partial^2}{\partial z^2}(xy^2 + z^3) \\
&= \frac{\partial}{\partial x}(y^2 + 0) + \frac{\partial}{\partial y}(2xy + 0) + \frac{\partial}{\partial z}(0 + 3z^2) \\
&= 0 + 2x + 6z = 2x + 6z
\end{aligned}$$

Exercise 2. Calculate the Laplacian of the following scalar fields: (click on the **green** letters for the solutions).

(a) $f(x, y, z) = 3x^3y^2z^3$ (b) $f(x, y, z) = \sqrt{xz} + y$

(c) $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ (d) $f(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$

Quiz Choose the Laplacian of $f(r) = \dfrac{1}{r^n}$ where $r = \sqrt{x^2 + y^2 + z^2}$.

(a) $-\dfrac{1}{r^{n+2}}$ (b) $\dfrac{n}{r^{n+2}}$

(c) $\dfrac{n(n-1)}{r^{n+2}}$ (d) $\dfrac{n(n+5)}{r^{n+2}}$

The equation $\boldsymbol{\nabla}^2 f = 0$ is called *Laplace's equation.* This is an important equation in science. From the above exercises and quiz we see that $f = \dfrac{1}{r}$ is a solution of Laplace's equation except at $r = 0$.

The Laplacian of a scalar field can also be written as follows:

$$\boxed{\boldsymbol{\nabla}^2 f = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} f}$$

i.e., as the **divergence of the gradient of f**. To see this consider

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{\nabla} f &= \boldsymbol{\nabla}\cdot\left(\frac{\partial f}{\partial x}\boldsymbol{i}+\frac{\partial f}{\partial y}\boldsymbol{j}+\frac{\partial f}{\partial z}\boldsymbol{k}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)\\
&= \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2} = \boldsymbol{\nabla}^2 f
\end{aligned}$$

EXERCISE 3. Find the Laplacian of the scalar fields f whose gradients $\boldsymbol{\nabla} f$ are given below (click on the **green** letters for the solutions).

(a) $\boldsymbol{\nabla} f = 2xz\boldsymbol{i} + x^2\boldsymbol{k}$

(b) $\boldsymbol{\nabla} f = \dfrac{1}{yz}\boldsymbol{i} - \dfrac{x}{y^2 z}\boldsymbol{j} - \dfrac{x}{yz^2}\boldsymbol{k}$

(c) $\boldsymbol{\nabla} f = \mathrm{e}^z\boldsymbol{i} + y\boldsymbol{j} + x\mathrm{e}^z\boldsymbol{k}$

(d) $\boldsymbol{\nabla} f = \dfrac{1}{x}\boldsymbol{i} + \dfrac{1}{y}\boldsymbol{j} + \dfrac{1}{z}\boldsymbol{k}$

3. The Laplacian of a Product of Fields

If a field may be written as a product of two functions, then:

$$\boxed{\nabla^2(uv) = (\nabla^2 u)v + u\nabla^2 v + 2(\nabla u)\cdot(\nabla v)}$$

A proof of this is given at the end of this section.

Example 2 The Laplacian of $f(x,y,z) = (x+y+z)(x-2z)$ may be directly calculated from the above rule

$$\begin{aligned}\nabla^2 f(x,y,z) &= (\nabla^2(x+y+z))(x-2z) + (x+y+z)\nabla^2(x-2z)\\ &\quad +2\nabla(x+y+z)\cdot\nabla(x-2z)\end{aligned}$$

Now $\nabla^2(x+y+z) = 0$ and $\nabla^2(x-2z) = 0$ so the first line on the right hand side vanishes.
To calculate the second line we note that

$$\nabla(x+y+z) = \frac{\partial(x+y+z)}{\partial x}\boldsymbol{i} + \frac{\partial(x+y+z)}{\partial y}\boldsymbol{j} + \frac{\partial(x+y+z)}{\partial z}\boldsymbol{k} = \boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$$

and

$$\nabla(x-2z) = \frac{\partial(x-2z)}{\partial x}\boldsymbol{i} + \frac{\partial(x-2z)}{\partial y}\boldsymbol{j} + \frac{\partial(x-2z)}{\partial z}\boldsymbol{k} = \boldsymbol{i} - 2\boldsymbol{k}$$

and taking their scalar product we obtain

$$\begin{aligned}\nabla^2 f(x,y,z) &= 0 + 2\nabla(x+y+z)\cdot\nabla(x-2z)\\ &= 2(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})\cdot(\boldsymbol{i}-2\boldsymbol{k}) = 2(1+0-2)\\ &= -2.\end{aligned}$$

This example may be checked by expanding $(x+y+z)(x-2z)$ and directly calculating the Laplacian.

Exercise 4. Use this rule to calculate the Laplacian of the scalar fields given below (click on the **green** letters for the solutions).

(a) $(2x-5y+z)(x-3y+z)$ (b) $(x^2-y)(x+z)$

(c) $(y-z)(x^2+y^2+z^2)$ (d) $x\sqrt{x^2+y^2+z^2}$

Proof that $\nabla^2(uv) = (\nabla^2 u)v + u\nabla^2 v + 2(\nabla u)\cdot(\nabla v)$.

By definition $\nabla^2(uv) = \frac{\partial^2}{\partial x^2}(uv) + \frac{\partial^2}{\partial y^2}(uv) + \frac{\partial^2}{\partial z^2}(uv)$. Consider therefore:

$$\begin{aligned}\frac{\partial^2}{\partial x^2}(uv) &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(uv)\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}v + u\frac{\partial v}{\partial x}\right)\\ &= \frac{\partial^2 u}{\partial x^2}v + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2}\\ &= \frac{\partial^2 u}{\partial x^2}v + 2\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2}\end{aligned}$$

where the **product rule** was repeatedly used.

By symmetry we also have: $\frac{\partial^2}{\partial y^2}(uv) = \frac{\partial^2 u}{\partial y^2}v + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + u\frac{\partial^2 v}{\partial y^2}$ and $\frac{\partial^2}{\partial z^2}(uv) = \frac{\partial^2 u}{\partial z^2}v + 2\frac{\partial u}{\partial z}\frac{\partial v}{\partial z} + u\frac{\partial^2 v}{\partial z^2}$. Adding these results yields the desired result.

4. The Laplacian and Vector Fields

If the scalar Laplacian operator is applied to a vector field, it acts on each component in turn and generates a vector field.

Example 3 The Laplacian of $\boldsymbol{F}(x,y,z) = 3z^2\boldsymbol{i} + xyz\boldsymbol{j} + x^2z^2\boldsymbol{k}$ is:

$$\boldsymbol{\nabla}^2\boldsymbol{F}(x,y,z) = \boldsymbol{\nabla}^2(3z^2)\boldsymbol{i} + \boldsymbol{\nabla}^2(xyz)\boldsymbol{j} + \boldsymbol{\nabla}^2(x^2z^2)\boldsymbol{k}$$

Calculating the components in turn we find:

$$\begin{aligned}
\boldsymbol{\nabla}^2(3z^2) &= \frac{\partial^2}{\partial x^2}(3z^2) + \frac{\partial^2}{\partial y^2}(3z^2) + \frac{\partial^2}{\partial z^2}(3z^2) = 0+0+6 = 6 \\
\boldsymbol{\nabla}^2(xyz) &= \frac{\partial^2}{\partial x^2}(xyz) + \frac{\partial^2}{\partial y^2}(xyz) + \frac{\partial^2}{\partial z^2}(xyz) = 0+0+0 = 0 \\
\boldsymbol{\nabla}^2(x^2z^2) &= \frac{\partial^2}{\partial x^2}(x^2z^2) + \frac{\partial^2}{\partial y^2}(x^2z^2) + \frac{\partial^2}{\partial z^2}(x^2z^2) = 2z^2+0+2x^2
\end{aligned}$$

So the Laplacian of $\boldsymbol{F}$ is:

$$\boldsymbol{\nabla}^2\boldsymbol{F} = 6\boldsymbol{i} + 0\boldsymbol{j} + (2z^2+2x^2)\boldsymbol{k} = 6\boldsymbol{i} + 2(x^2+z^2)\boldsymbol{k}$$

Quiz Select from the answers below the Laplacian of the vector field $\boldsymbol{F} = x^3y\boldsymbol{i} + \ln(z)\boldsymbol{j} + \ln(xy)\boldsymbol{k}$.

(a) $6xy\boldsymbol{i}$

(b) $6xy\boldsymbol{i} - \dfrac{1}{z^2}\boldsymbol{j} - \dfrac{x^2+y^2}{x^2y^2}\boldsymbol{k}$

(c) $3x\boldsymbol{i} + \dfrac{1}{z^2}\boldsymbol{j} - \left(\dfrac{1}{x^2} - \dfrac{1}{y^2}\right)\boldsymbol{k}$

(d) $y^3\boldsymbol{i} + x^2\boldsymbol{j} - \dfrac{y^2-x^2}{z^2}\boldsymbol{k}$

Quiz Choose the Laplacian of $\boldsymbol{F} = 3x^2z\boldsymbol{i} - \sin(\pi y)\boldsymbol{j} + \ln(2x^3)\boldsymbol{k}$ at the point $(1, -2, 1)$?

(a) 0 (b) $6\boldsymbol{i} - 3\boldsymbol{k}$ (c) $3\boldsymbol{i} - 3\boldsymbol{k}$ (d) $6\boldsymbol{i} - \pi^2\boldsymbol{j} - 6\boldsymbol{k}$

Quiz Which of the following choices is the Laplacian of the vector field $\boldsymbol{F} = \ln(y)\boldsymbol{i} + z^2\boldsymbol{j} - \sin(2\pi x)\boldsymbol{k}$ at $(1, 1, \pi)$?

(a) $-\boldsymbol{i} + 2\boldsymbol{j}$ (b) 0 (c) $4\boldsymbol{j} + 4\pi^2\boldsymbol{k}$ (d) $-\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

5. Final Quiz

Begin Quiz Choose the solutions from the options given.

1. Choose the Laplacian of $f(\boldsymbol{r}) = 5x^3y^4z^2$.

(a) $30xy^4z^2 + 60x^3y^2z^2 + 10x^3y^4$ (b) $30x + 20y^2 + 10$

(c) $30xy^4z^2 + 75x^3y^2z^2 + 15x^3y^4$ (d) $30xy^4z^2 + 12x^3y^2z^2 + 15x^3y^4$

2. Choose the Laplacian of $f(x, y, z) = \ln(r)$ where $r = \sqrt{x^2 + y^2 + z^2}$.

(a) 0 (b) $\dfrac{2}{r^2}$ (c) $\dfrac{1}{2r^2}$ (d) $\dfrac{1}{r^2}$

3. Select the Laplacian of $\boldsymbol{F} = x^3\boldsymbol{i} + 7y\boldsymbol{j} - 3\sin(2y)\boldsymbol{k}$.

(a) $6x\boldsymbol{i}$ (b) $6x\boldsymbol{i} + 12\sin(2y)\boldsymbol{k}$

(c) $6x\boldsymbol{i} + 3\sin(2y)\boldsymbol{k}$ (d) $6x\boldsymbol{i} - 12\sin(2y)\boldsymbol{k}$

End Quiz Score: Correct

Solutions to Exercises

Exercise 1(a)

To find the **gradient** of the scalar field $f = x^2y - z$, we need the partial derivatives:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(x^2y - z) = 2x^{2-1} \times y = 2xy\,,\\ \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(x^2y - z) = x^2 \times y^{1-1} = x^2\,,\\ \frac{\partial f}{\partial z} &= \frac{\partial}{\partial z}(x^2y - z) = 0 - z^{1-1} = -1\,.\end{aligned}$$

Therefore the gradient of $f = x^2y - z$ is

$$\begin{aligned}\boldsymbol{\nabla} f(x,y,z) &= \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}\\ &= 2xy\boldsymbol{i} + x^2\boldsymbol{j} - \boldsymbol{k}\,.\end{aligned}$$

Click on the **green** square to return

□

Exercise 1(b)

To find the divergence of the vector field $\boldsymbol{F} = x\boldsymbol{i} - xy\boldsymbol{j} + z^2\boldsymbol{k}$, we recognise that its components are

$$F_1 = x\,, \qquad F_2 = -xy\,, \qquad F_3 = z^2\,,$$

So the divergence is the scalar expression

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{F} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \\
&= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-xy) + \frac{\partial}{\partial z}(z^2) \\
&= x^{1-1} - x \times y^{1-1} + 2 \times z^{2-1} \\
&= 1 - x + 2z\,.
\end{aligned}$$

Click on the **green** square to return $\square$

Exercise 1(c)
The *curl* of the vector field $\boldsymbol{F}$ whose components are

$$F_1 = x\,, \qquad F_2 = -xy\,, \qquad F_3 = z^2\,,$$

is given by the vector expression:

$$\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{F} &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\boldsymbol{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\boldsymbol{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k} \\
&= \left(\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(-xy)\right)\boldsymbol{i} - \left(\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial z}(x)\right)\boldsymbol{j} \\
&\qquad + \left(\frac{\partial}{\partial x}(-xy) - \frac{\partial}{\partial y}(x)\right)\boldsymbol{k} \\
&= (0+0)\,\boldsymbol{i} - (0-0)\,\boldsymbol{j} + (-y-0)\,\boldsymbol{k} \\
&= -y\boldsymbol{k}\,.
\end{aligned}$$

Click on the **green** square to return □

Exercise 1(d)

To subtract the curl of the vector $\boldsymbol{F} = x\boldsymbol{i} - xy\boldsymbol{j} + z^2\boldsymbol{k}$ from the gradient of the scalar field $f = x^2y - z$

$$\boldsymbol{\nabla} f - \boldsymbol{\nabla} \times \boldsymbol{F}\,.$$

we use the results of Exercise 1a and Exercise 1c, where it was found that

$$\boldsymbol{\nabla} f = 2xy\boldsymbol{i} + x^2\boldsymbol{j} - \boldsymbol{k} \quad \text{and} \quad \boldsymbol{\nabla} \times \boldsymbol{F} = -y\boldsymbol{k}\,.$$

Therefore the difference of these two vectors is

$$\begin{aligned} \boldsymbol{\nabla} f - \boldsymbol{\nabla} \times \boldsymbol{F} &= 2xy\boldsymbol{i} + x^2\boldsymbol{j} - \boldsymbol{k} - (-y)\boldsymbol{k} \\ &= 2xy\boldsymbol{i} + x^2\boldsymbol{j} - (1-y)\boldsymbol{k}\,. \end{aligned}$$

Click on the **green** square to return □

Exercise 2(a)
The Laplacian of the scalar field $f = 3x^3y^2z^3$ is:

$$\begin{aligned}
\nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\
&= \frac{\partial^2}{\partial x^2}(3x^3y^2z^3) + \frac{\partial^2}{\partial y^2}(3x^3y^2z^3) + \frac{\partial^2}{\partial z^2}(3x^3y^2z^3) \\
&= \frac{\partial}{\partial x}(9x^2y^2z^3) + \frac{\partial}{\partial y}(6x^3yz^3) + \frac{\partial}{\partial z}(9x^3y^2z^2) \\
&= 18xy^2z^3 + 6x^3z^3 + 18x^3y^2z\,.
\end{aligned}$$

Extracting common factors, the scalar $\nabla^2 f$ can also be written as

$$\begin{aligned}
\nabla^2 f &= 6xz\left(3y^2z^2 + x^2z^2 + 3x^2y^2\right) \\
&= 6xz\left(3y^2(z^2 + x^2) + x^2z^2\right)\,.
\end{aligned}$$

Click on the **green** square to return

Exercise 2(b)

The Laplacian of the scalar field $f = \sqrt{xz} + y = x^{1/2}z^{1/2} + y$ is:

$$\begin{aligned}
\boldsymbol{\nabla}^2 f &= \frac{\partial^2}{\partial x^2}\left(x^{\frac{1}{2}}z^{\frac{1}{2}}+y\right)+\frac{\partial^2}{\partial y^2}\left(x^{\frac{1}{2}}z^{\frac{1}{2}}+y\right)+\frac{\partial^2}{\partial z^2}\left(x^{\frac{1}{2}}z^{\frac{1}{2}}+y\right)\\
&= \frac{\partial}{\partial x}\left(\frac{1}{2}x^{(\frac{1}{2}-1)}z^{\frac{1}{2}}\right)+\frac{\partial}{\partial y}(1)+\frac{\partial}{\partial z}\left(\frac{1}{2}x^{\frac{1}{2}}z^{(\frac{1}{2}-1)}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{1}{2}x^{-\frac{1}{2}}z^{\frac{1}{2}}\right)+0+\frac{\partial}{\partial z}\left(\frac{1}{2}x^{\frac{1}{2}}z^{-\frac{1}{2}}\right)\\
&= \frac{1}{2}(-\frac{1}{2})x^{(-\frac{1}{2}-1)}z^{\frac{1}{2}}+\frac{1}{2}(-\frac{1}{2})x^{\frac{1}{2}}z^{(-\frac{1}{2}-1)}\\
&= -\frac{1}{4}x^{-\frac{3}{2}}z^{\frac{1}{2}}-\frac{1}{4}x^{\frac{1}{2}}z^{-\frac{3}{2}}\,.
\end{aligned}$$

This result may be rewritten as

$$\boldsymbol{\nabla}^2 f = -\frac{1}{4}x^{\frac{1}{2}}z^{\frac{1}{2}}\left(x^{-2}+z^{-2}\right) = -\frac{1}{4}\sqrt{xz}\left(\frac{1}{x^2}+\frac{1}{z^2}\right)\,.$$

Click on the **green** square to return □

Exercise 2(c) To calculate $\boldsymbol{\nabla}^2\sqrt{x^2+y^2+z^2}$, define $u = x^2+y^2+z^2$, so $f = u^{1/2}$. From the chain rule we have

$$\frac{\partial f}{\partial x} = \frac{\partial u^{\frac{1}{2}}}{\partial u} \times \frac{\partial u}{\partial x} = \frac{1}{2}u^{(\frac{1}{2}-1)} \times 2x = xu^{-\frac{1}{2}}\,.$$

Therefore the second derivative is (from the product and chain rules):

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(x) \times u^{-\frac{1}{2}} + x \times \frac{\partial u^{-\frac{1}{2}}}{\partial u} \times \frac{\partial u}{\partial x} = u^{-\frac{1}{2}} - x^2 u^{-\frac{3}{2}}\,.$$

Since f and u are symmetric in x, y and z we have

$$\frac{\partial^2 f}{\partial y^2} = u^{-\frac{1}{2}} - y^2 u^{-\frac{3}{2}} \qquad \text{and} \qquad \frac{\partial^2 f}{\partial z^2} = u^{-\frac{1}{2}} - z^2 u^{-\frac{3}{2}}\,.$$

Adding these results and using $x^2+y^2+z^2 = u$ yields

$$\begin{aligned} \boldsymbol{\nabla}^2 f &= \left(u^{-\frac{1}{2}} - x^2u^{-\frac{3}{2}}\right) + \left(u^{-\frac{1}{2}} - y^2u^{-\frac{3}{2}}\right) + \left(u^{-\frac{1}{2}} - z^2u^{-\frac{3}{2}}\right) \\ &= 3u^{-\frac{1}{2}} - (x^2+y^2+z^2)u^{-\frac{3}{2}} = 2u^{-\frac{1}{2}} = \frac{2}{\sqrt{x^2+y^2+z^2}}\,. \end{aligned}$$

□

Click on the **green** square to return

Exercise 2(d) To find the Laplacian of $f = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$, we again define $u = x^2 + y^2 + z^2$. From the chain rule

$$\frac{\partial f}{\partial x} = \frac{\partial u^{-\frac{1}{2}}}{\partial u} \times \frac{\partial u}{\partial x} = -\frac{1}{2} u^{(-\frac{1}{2}-1)} \times 2x = -xu^{-\frac{3}{2}} .$$

Thus the second order derivative is:

$$\frac{\partial^2 f}{\partial x^2} = - \frac{\partial}{\partial x}(x) \times u^{-\frac{3}{2}} - x \times \frac{\partial u^{-\frac{3}{2}}}{\partial u} \times \frac{\partial u}{\partial x} = -u^{-\frac{3}{2}} + 3x^2 u^{-\frac{5}{2}} .$$

Due to the symmetry under interchange of x, y and z:

$$\frac{\partial^2 f}{\partial y^2} = -u^{-\frac{3}{2}} + 3y^2 u^{-\frac{5}{2}} , \qquad \frac{\partial^2 f}{\partial z^2} = -u^{-\frac{3}{2}} + 3z^2 u^{-\frac{5}{2}} .$$

Therefore we find that the Laplacian of f vanishes:

$$\begin{aligned} \boldsymbol{\nabla}^2 f &= \left(-u^{-\frac{3}{2}} + 3x^2 u^{-\frac{5}{2}}\right) + \left(-u^{-\frac{3}{2}} + y^2 u^{-\frac{5}{2}}\right) + \left(-u^{-\frac{3}{2}} + 3z^2 u^{-\frac{5}{2}}\right) \\ &= -3u^{-\frac{3}{2}} + 3(x^2 + y^2 + z^2) u^{-\frac{5}{2}} = -3u^{-\frac{3}{2}} + 3u^{-\frac{3}{2}} = 0 . \end{aligned}$$

Click on the **green** square to return □

Exercise 3(a)
If the gradient of the scalar function f is $\boldsymbol{\nabla} f = 2xz\boldsymbol{i} + x^2\boldsymbol{k}$, then the Laplacian of f is given by the divergence of this vector

$$\boldsymbol{\nabla}^2 f = \mathrm{div}\,(\boldsymbol{\nabla} f)\ .$$

Therefore the Laplacian of f is

$$\begin{aligned}
\boldsymbol{\nabla}^2 f &= \mathrm{div}\left(2xz\boldsymbol{i} + x^2\boldsymbol{k}\right) \\
&= \frac{\partial}{\partial x}(2xz) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(x^2) \\
&= 2z + 0 + 0 \\
&= 2z\,.
\end{aligned}$$

Click on the **green** square to return $\square$

Exercise 3(b)

If a scalar field f has gradient $\boldsymbol{\nabla} f = \frac{1}{yz}\boldsymbol{i} - \frac{x}{y^2 z}\boldsymbol{j} - \frac{x}{yz^2}\boldsymbol{k}$, its Laplacian is given by the divergence

$$\boldsymbol{\nabla}^2 f = \mathrm{div}\,(\boldsymbol{\nabla} f)\ .$$

Using this formula we can find $\boldsymbol{\nabla}^2 f$ as follows

$$\begin{aligned}
\boldsymbol{\nabla}^2 f &= \mathrm{div}\left(\frac{1}{yz}\boldsymbol{i} - \frac{x}{y^2 z}\boldsymbol{j} - \frac{x}{yz^2}\boldsymbol{k}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{1}{yz}\right) + \frac{\partial}{\partial y}\left(-\frac{x}{y^2 z}\right) + \frac{\partial}{\partial z}\left(-\frac{x}{yz^2}\right) \\
&= (0) - (-2) \times \frac{x}{y^3 z} - (-2) \times \frac{x}{yz^3} \\
&= 2x\,\frac{y^2 + z^2}{y^3 z^3}\ .
\end{aligned}$$

Click on the **green** square to return

□

Exercise 3(c)
As above if the gradient of f is

$$\boldsymbol{\nabla} f = \mathrm{e}^z \boldsymbol{i} + y\boldsymbol{j} + x\mathrm{e}^z \boldsymbol{k}$$

its Laplacian is the scalar divergence of the vector $\boldsymbol{\nabla} f$:

$$\boldsymbol{\nabla}^2 f = \mathrm{div}\,(\boldsymbol{\nabla} f)\ .$$

Therefore

$$\begin{aligned}
\boldsymbol{\nabla}^2 f &= \mathrm{div}\,(\mathrm{e}^z \boldsymbol{i} + y\boldsymbol{j} + x\mathrm{e}^z \boldsymbol{k}) \\
&= \frac{\partial}{\partial x}\,(\mathrm{e}^z) + \frac{\partial}{\partial y}\,(y) + \frac{\partial}{\partial z}\,(x\mathrm{e}^z) \\
&= (0) + 1 + x\mathrm{e}^z \\
&= 1 + x\mathrm{e}^z\,.
\end{aligned}$$

Click on the **green** square to return □

Exercise 3(d)

If $\boldsymbol{\nabla} f = \frac{1}{x}\boldsymbol{i} + \frac{1}{y}\boldsymbol{j} + \frac{1}{z}\boldsymbol{k}$, then the Laplacian of f is the divergence of the gradient of f:

$$\boldsymbol{\nabla}^2 f = \mathrm{div}\,(\boldsymbol{\nabla} f)\ .$$

Therefore

$$\begin{aligned}
\boldsymbol{\nabla}^2 f &= \mathrm{div}\left(\frac{1}{x}\boldsymbol{i} + \frac{1}{y}\boldsymbol{j} + \frac{1}{z}\boldsymbol{k}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{1}{x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{y}\right) + \frac{\partial}{\partial z}\left(\frac{1}{z}\right) \\
&= -\frac{1}{x^2} - \frac{1}{y^2} - \frac{1}{z^2}\,.
\end{aligned}$$

This may also be written as follows

$$\boldsymbol{\nabla}^2 f = -\frac{x^2y^2 + z^2x^2 + y^2z^2}{x^2y^2z^2}\,.$$

Click on the **green** square to return $\square$

Exercise 4(a)
To find the **Laplacian** of $f = (2x - 5y + z)(x - 3y + z)$, we use

$$\boldsymbol{\nabla}^2(uv) = (\boldsymbol{\nabla}^2 u)v + u\boldsymbol{\nabla}^2 v + 2(\boldsymbol{\nabla} u)\cdot(\boldsymbol{\nabla} v)$$

with $u = 2x - 5y + z$ and $v = x - 3y + z$. This implies

$$\begin{aligned}\boldsymbol{\nabla} u &= 2\boldsymbol{i} - 5\boldsymbol{j} + \boldsymbol{k} \\ \boldsymbol{\nabla} v &= \boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}\end{aligned}$$

and so

$$\boldsymbol{\nabla}^2 u = \boldsymbol{\nabla}^2 v = 0$$

The above rule therefore gives:

$$\begin{aligned}\boldsymbol{\nabla}^2 f = 2\boldsymbol{\nabla} u\cdot\boldsymbol{\nabla} v &= 0 + 0 + 2(2\boldsymbol{i} - 5\boldsymbol{j} + \boldsymbol{k})\cdot(\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}) \\ &= 2(2 + 15 + 1) \\ &= 36\,.\end{aligned}$$

Click on the **green** square to return

□

Exercise 4(b)

To find the **Laplacian** of $f = (x^2 - y)(x + z)$, we use

$$\boldsymbol{\nabla}^2(uv) = (\boldsymbol{\nabla}^2 u)v + u\boldsymbol{\nabla}^2 v + 2(\boldsymbol{\nabla} u) \cdot (\boldsymbol{\nabla} v)$$

with $u = x^2 - y$ and $v = x + z$. This implies

$$\begin{aligned} \boldsymbol{\nabla} u &= 2x\boldsymbol{i} - \boldsymbol{j} \\ \boldsymbol{\nabla} v &= \boldsymbol{i} + \boldsymbol{k} \end{aligned}$$

and so

$$\boldsymbol{\nabla}^2 u = \boldsymbol{\nabla} \cdot 2x\boldsymbol{i} = \frac{\partial(2x)}{\partial x} = 2 \quad \text{and} \quad \boldsymbol{\nabla}^2 v = 0$$

The above rule therefore gives:

$$\begin{aligned} \boldsymbol{\nabla}^2 f &= 2(x+z) + 0 + 2(2\boldsymbol{i} - \boldsymbol{j}) \cdot (\boldsymbol{i} + \boldsymbol{k}) \\ &= 2(x+z) + 2(2) \\ &= 2(x+z+2)\,. \end{aligned}$$

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Exercise 4(c)
To find the **Laplacian** of $f = (y - z)(x^2 + y^2 + z^2)$, we use

$$\boldsymbol{\nabla}^2(uv) = (\boldsymbol{\nabla}^2 u)v + u\boldsymbol{\nabla}^2 v + 2(\boldsymbol{\nabla} u)\cdot(\boldsymbol{\nabla} v)$$

with $u = y - z$ and $v = x^2 + y^2 + z^2$. This implies

$$\boldsymbol{\nabla} u = \boldsymbol{j} - \boldsymbol{k} \quad \text{and} \quad \boldsymbol{\nabla}^2 u = 0$$

while

$$\begin{aligned} \boldsymbol{\nabla} v &= 2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k} \\ \boldsymbol{\nabla}^2 u &= 2 + 2 + 2 = 6 \end{aligned}$$

The above rule therefore gives:

$$\begin{aligned} \boldsymbol{\nabla}^2 f &= 0 + (y - z)\times 6 + 2(\boldsymbol{j} - \boldsymbol{k})\cdot(2\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}) \\ &= 6 + 2(0 + 2 - 2) \\ &= 6\,. \end{aligned}$$

Click on the **green** square to return $\square$

Exercise 4(d) To find the **Laplacian** of $f = x\sqrt{x^2+y^2+z^2}$, we use

$$\boldsymbol{\nabla}^2(uv) = (\boldsymbol{\nabla}^2 u)v + u\boldsymbol{\nabla}^2 v + 2(\boldsymbol{\nabla} u)\cdot(\boldsymbol{\nabla} v)$$

with $u = x$ and $v = \sqrt{x^2+y^2+z^2}$. Therefore $\boldsymbol{\nabla} u = \boldsymbol{i}$ and $\boldsymbol{\nabla}^2 u = 0$. In Exercise 2(c) we saw $\boldsymbol{\nabla}^2 v = -2/\sqrt{x^2+y^2+z^2}$. To find $\boldsymbol{\nabla} v$ use the **chain rule**: $\dfrac{\partial}{\partial x}(x^2+y^2+z^2)^{-\frac{1}{2}} = 2x\times(-\frac{1}{2})\times(x^2+y^2+z^2)^{-\frac{3}{2}}$ and similarly for the other partial derivatives. Thus

$$\boldsymbol{\nabla} v = -\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})$$

The above rule therefore gives:

$$\begin{aligned}\boldsymbol{\nabla}^2 f &= 0 + x\times\frac{-2}{\sqrt{x^2+y^2+z^2}} + 2\boldsymbol{i}\cdot\frac{-1}{(x^2+y^2+z^2)^{\frac{3}{2}}}(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}) \\ &= -\frac{2x}{\sqrt{x^2+y^2+z^2}} - \frac{2x^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}\,.\end{aligned}$$

Click on the **green** square to return □

Solutions to Quizzes

Solution to Quiz: The first and the second order partial derivatives of $f = r^{-n}\,, r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$ with respect to the variable x read:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial r^{-n}}{\partial r} \times \frac{\partial r}{\partial x} = -nr^{-n-1} \times \frac{x}{r} = -n\frac{x}{r^{n+2}}\,. \\
\frac{\partial^2 f}{\partial x^2} &= -n\frac{1}{r^{n+2}}\frac{\partial x}{\partial x} - xn \times \frac{\partial}{\partial r}\left(\frac{1}{r^{n+2}}\right) \times \frac{\partial r}{\partial x} \\
&= -n\frac{1}{r^{n+2}} + n(n+2)\frac{x^2}{r^{n+4}}\,.
\end{aligned}$$

Due to the symmetry under interchange of x, y and z we have also

$$\frac{\partial^2 f}{\partial y^2} = -n\frac{1}{r^{n+2}} + n(n+2)\frac{y^2}{r^{n+4}}\,, \quad \frac{\partial^2 f}{\partial z^2} = -n\frac{1}{r^{n+2}} + n(n+2)\frac{z^2}{r^{n+4}}\,.$$

Adding these second order derivatives yields the Laplacian:

$$\boldsymbol{\nabla}^2 f = -\frac{3n}{r^{n+2}} + n(n+2)\frac{x^2 + y^2 + z^2}{r^{n+4}} = \frac{n(n-1)}{r^{n+2}}\,.$$

End Quiz

Solution to Quiz:
The Laplacian of the vector field $\boldsymbol{F} = x^3y\boldsymbol{i} + \ln(z)\boldsymbol{j} + \ln(xy)\boldsymbol{k}$ is a vector $\boldsymbol{\nabla}^2\boldsymbol{F}$ whose $\boldsymbol{i}\,,\boldsymbol{j}$ components are correspondingly

$$\begin{aligned}
\boldsymbol{\nabla}^2(x^3y) &= \frac{\partial^2}{\partial x^2}(x^3y) + \frac{\partial^2}{\partial y^2}(x^3y) + \frac{\partial^2}{\partial z^2}(x^3y) = 6xy\,, \\
\boldsymbol{\nabla}^2(\ln(z)) &= \frac{\partial^2}{\partial z^2}(\ln(z)) = \frac{\partial}{\partial z}\left(\frac{1}{z}\right) = -\frac{1}{z^2}\,,
\end{aligned}$$

while the $\boldsymbol{k}$ component is

$$\boldsymbol{\nabla}^2(\ln(xy)) = \boldsymbol{\nabla}^2(\ln(x) + \ln(y)) = -\frac{1}{x^2} - \frac{1}{y^2} = -\frac{x^2+y^2}{x^2y^2}\,.$$

So the Laplacian of $\boldsymbol{F}$ is:

$$\boldsymbol{\nabla}^2\boldsymbol{F} = 6xy\boldsymbol{i} - \frac{1}{z^2}\boldsymbol{j} - \frac{x^2+y^2}{x^2y^2}\boldsymbol{k}\,.$$

End Quiz

Solution to Quiz:
The Laplacian of the vector field $\boldsymbol{F} = 3x^2 z\boldsymbol{i} - \sin(\pi y)\boldsymbol{j} + \ln(2x^3)\boldsymbol{k}$ is a vector $\boldsymbol{\nabla}^2\boldsymbol{F}$ whose $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ components are correspondingly

$$\begin{aligned}
\boldsymbol{\nabla}^2\left(3x^2 z\right) &= \frac{\partial^2}{\partial x^2}(3x^2 z) + \frac{\partial^2}{\partial y^2}(3x^2 z) + \frac{\partial^2}{\partial z^2}(3x^2 z) = 6z\,, \\
\boldsymbol{\nabla}^2\left(-\sin(\pi y)\right) &= -\frac{\partial^2}{\partial y^2}(\sin(\pi y)) = -\frac{\partial}{\partial y}(\pi\cos(\pi y)) = \pi^2\sin(\pi y)\,, \\
\boldsymbol{\nabla}^2\left(\ln(2x^3)\right) &= \frac{\partial^2}{\partial x^2}(\ln(2x^3)) = \frac{\partial^2}{\partial x^2}(\ln(2) + 3\ln(x)) = -3\frac{1}{x^2}\,,
\end{aligned}$$

Therefore the Laplacian of $\boldsymbol{F}$ is

$$\boldsymbol{\nabla}^2\boldsymbol{F} = 6z\boldsymbol{i} + \pi^2\sin(\pi y)\boldsymbol{j} - 3\frac{1}{x^2}\boldsymbol{k}\,,$$

and evaluating it at the point $(1, -2, 1)$ we get

$$\boldsymbol{\nabla}^2\boldsymbol{F} = 6\boldsymbol{i} + \pi^2\sin(-2\pi)\boldsymbol{j} - 3\boldsymbol{k} = 6\boldsymbol{i} - 3\boldsymbol{k}\,.$$

End Quiz

Solution to Quiz:
The Laplacian of $\boldsymbol{F} = \ln(y)\boldsymbol{i} + z^2\boldsymbol{j} - \sin(2\pi x)\boldsymbol{k}$ at $(1, 1, \pi)$ is the vector whose $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ components are in turn given by:

$$\begin{aligned}
\nabla^2 \ln(y) &= \frac{\partial^2 \ln(y)}{\partial x^2} + \frac{\partial^2 \ln(y)}{\partial y^2} + \frac{\partial^2 \ln(y)}{\partial z^2} = 0 - \frac{1}{y^2} + 0\,, \\
\nabla^2 z^2 &= \frac{\partial^2 z^2}{\partial x^2} + \frac{\partial^2 z^2}{\partial y^2} + \frac{\partial^2 z^2}{\partial z^2} = 0 + 0 + 2\,, \\
\nabla^2 \sin(2\pi x) &= \frac{\partial^2 \sin(2\pi x)}{\partial x^2} + \frac{\partial^2 \sin(2\pi x)}{\partial y^2} + \frac{\partial^2 \sin(2\pi x)}{\partial z^2} \\
&= -4\pi^2 \sin(2\pi x) + 0 + 0\,,
\end{aligned}$$

Therefore we have

$$\nabla^2 \boldsymbol{F} = -\frac{1}{y^2}\boldsymbol{i} + 2\boldsymbol{j} - 4\pi^2 \sin(2\pi x)\boldsymbol{k}$$

Since $\sin(2\pi) = 0$, we find at $(1, 1, \pi)$ that $\nabla^2 \boldsymbol{F} = -\boldsymbol{i} + 2\boldsymbol{j}$.

End Quiz